# Dirac operators and the calculation of the Connes metric on arbitrary (infinite) graphs

**Manfred Requardt**

Institut für Theoretische Physik, Universität Göttingen, Bunsenstrasse 9, 37073 Göttingen, Germany

E-mail: requardt@theorie.physik.uni-goettingen.de

**Abstract**
As an outgrowth of our investigation of non-regular spaces within the context of quantum gravity and non-commutative geometry, we develop a graph Hilbert space framework on arbitrary (infinite) graphs and use it to study spectral properties of *graph Laplacians* and *graph Dirac operators*. We define a *spectral triplet* sharing most of the properties of what Connes calls a *spectral triple*. With the help of this scheme we derive an explicit expression for the *Connes-distance function* on general directed or undirected graphs. We derive a series of *a priori* estimates and calculate it for a variety of examples of graphs. As a possibly interesting side, we show that the natural setting for approaching such problems may be the framework of (*non*)*linear programming* or *optimization*. We compare our results (arrived at within our particular framework) with those of other authors and show that the seeming differences depend on the use of different graph geometries and/or Dirac operators.

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## 1. Introduction

In recent years, we started a programme to reconstruct continuum physics and/or mathematics from an underlying more primordial and basically discrete theory living on the Planck-scale (cf [1–4, 37]). As a sort of a 'spin-off' various problems of a more mathematical and technical flavour emerged which may have an interest of their own. *Discrete differential geometric* concepts were dealt in [1], the theory of *random graphs* was a central theme of [2], topics of *dimension theory* and *fractal geometry* were addressed in [4], and *lump-spaces* and *random metrics* were treated in [37].

If one wants to recover the usual (differential) operators (or more generally, the concepts of standard functional analysis), being used in ordinary continuum physics and mathematics, by some sort of limiting process from their discrete protoforms, which live, for their part, on

a relatively disordered discrete background such as, say, a network, it is reasonable as a first step to analyse these discrete counterparts more closely. This will be one of our themes in the following with particular emphasis on discrete *Laplace* and *Dirac operators* on general *graphs*. In contrast to [16] we now also include arbitrary *directed graphs*. The main thrust goes, however, into an analysis of metrical concepts on discrete (non-commutative) spaces such as general graphs, being induced by *graph Dirac operators* and the *Connes-distance functional*.

We note that the *functional analysis on graphs* is of interest both in pure and in applied mathematics and also in various fields of (mathematical) physics. In general, discrete systems have an increasing interest of their own or serve as easier to analyse prototypes of their continuum counterparts. A few fields of applications are: *graph theory* in general, analysis on *(discrete) manifolds*, *lattice* or *discretized* versions of physical models in statistical mechanics and quantum field theory, *non-commutative geometry*, *networks*, *fractal geometry*, etc. From the widely scattered (mathematical and physical) literature we mention ( possibly) very few sources we are aware of. Some of them were of relevance for our own motivation, whereas we came across some others only recently (see, e.g., [5–14, 20, 33]). Further literature, e.g. [15], was pointed out to us by Mueller-Hoissen; the possible relevance of [17–19] was brought to our notice by an unknown referee. Last, but not the least, there is the vast field of *discretized quantum gravity* (see, e.g., [21, 22]). All these show that the sort of discrete functional analysis we are dealing with in the following is presently a very active field with a lot of different applications.

For the convenience of the reader we begin by compiling some concepts and tools dealing with graph Hilbert spaces which we then use to investigate the spectral properties of *graph Laplacians* and *Dirac operators*. In the next step we study and test concepts and ideas which arose in the context of *non-commutative geometry*. As we (and others) showed in preceding papers, networks and graphs may (or even should) be understood as examples of *non-commutative* spaces. Currently an interesting topic in this field is the investigation of certain *distance functionals* on 'nasty' or *non-standard* spaces and their mathematical or physical 'naturalness'. However, graphs carry a natural *metric structure* given by a *distance function* $d(x, y)$, with $x, y$ two *nodes* of the graph (see the following sections). This fact was already employed by us in [4] to develop dimensional concepts on graphs. Having Connes' concept of distance in non-commutative geometry in mind (cf ch VI of [5]), it is natural to try to compute it in model systems, which in our context means arbitrary graphs, and compare it with the already existing notion of graph distance mentioned above. (We note that the calculation of the Connes distance for general graphs turns out to be surprisingly complex and leads to perhaps unexpected connections to fields of mathematics, e.g. (*non*)*linear programming* or *optimization*; see the last section.)

Therefore, as one of the many possible applications of our formalism, we construct a protoform of what Connes calls a *spectral triple*, that is, a Hilbert space structure, a corresponding representation of a certain (function) algebra and a (in our framework) natural candidate for the so-called *Dirac operator* (not to be confused with the ordinary Dirac operator of the Dirac equation), which encodes certain properties of the *graph geometry*. This will be done in section 3.

In the last (and central) section, which deals with the distance concept derived from the spectral triplet, we will investigate this concept more closely as far as graphs and similar spaces are concerned. In this connection, some recent works should be mentioned, in which Connes' distance function was analysed in certain simple models, e.g. one-dimensional lattices [25–27]. These papers show that it is a touchy business to isolate 'the' appropriate Dirac operator (after all, different Dirac operators are expected to lead to different geometries!) and

that it is perhaps worthwhile to scrutinize the whole topic in a more systematic way. We show, in particular, that one may choose different Dirac operators on graphs (or rather, different types of graphs over the same node set) which may lead to different results for the corresponding Connes distance, for example.

The problem of finding suitable metrics on 'non-standard' spaces is a particularly interesting research topic of its own, presently pursued by quite a few people (see the papers by Rieffel [12] and Weaver [13] and the references therein). Another, earlier (and important), source is [15]. We recently extended the investigation of metric structures to the so-called *lump spaces* and *probabilistic metric spaces* (see [37, 38]) and employed it in the general context of quantum gravity (cf [2]).

*Remark*: When we wrote an earlier draft of this paper, we were unaware of the content of [15]. It happened only recently that we realized that various results we derived in connection with the Dirac operators and the Connes distance can already be found in [15] and we try to take care of this fact in the following. Both the technical approach and the motivation are, however, not entirely identical. The interested reader may consult the electronic version of our earlier draft [16].

The same applies to paper [10], where some of the Hilbert space methods were developed which we later rederived in [16], being unaware of the prior results. As a consequence we drop most of the technical steps leading to that part of our previous results overlapping with the corresponding parts in the above-mentioned papers and refer, for simplicity, to [16] where the interested reader can find more details.

## 2. A brief survey of differential and operator calculus on graphs and graph Hilbert spaces

We give a brief survey of certain concepts and tools needed in the following analysis. While our framework may deviate at various places from the more traditional one, employed in e.g. *algebraic graph theory* (see, e.g. [8, 9, 17]), this is mainly done for greater mathematical flexibility and generality and, on the other hand, possible physical applications (a case in point being the analysis of *non-commutative spaces*). Most of the technical tools which are not defined in detail in the following have been introduced in section 3 of [1].

### 2.1. Simple (symmetric) graphs

For simplicity, we assume the graph to be connected and *locally finite* (no elementary loops and multiedges, whereas these could easily be incorporated in the framework), i.e. each node (or vertex) is incident with only a finite number of edges (or bonds). To avoid operator domain problems we usually make an even stronger assumption that the *vertex* or *node degree*, $v_i$ ($i$ labelling the nodes), is globally bounded but this restriction is frequently not really necessary. Furthermore, it has turned out to be algebraically advantageous to identify an (undirected) labelled graph with a directed graph having two oppositely directed edges for each undirected edge, the directed edge, pointing from node $n_i$ to node $n_k$, being denoted by $d_{ik}$, the oppositely directed edge by $d_{ki}$ and the undirected (but orientable) edge by their superposition

$$b_{ik} = d_{ik} - d_{ki} = -b_{ki} \tag{1}$$

($d_{ik}$ and $d_{ki}$ are treated as independent basis vectors; cf [1, 16]).

As the elementary building blocks of our graph Hilbert spaces, we take $\{n_i\}$ and $\{d_{ik}\}$ as basis elements of a certain hierarchy of Hilbert spaces over, say, $\mathbb{C}$ with a scalar product induced by

$$(n_i|n_k) = \delta_{ik} \qquad (d_{ik}|d_{lm}) = \delta_{il}\delta_{km} \tag{2}$$

This definition implies $(b_{ik}|b_{ik}) = 2$.

**Definition 2.1** (vertex, edge Hilbert space). *The Hilbert spaces $H_0, H_1^a$ (a for antisymmetric) and $H_1$ consist of the formal sums*

$$f := \sum f_i n_i \qquad g := \sum g_{ik} d_{ik} \qquad with \quad g_{ik} = -g_{ki} \quad and \quad g' := \sum g_{ik} d_{ik} \tag{3}$$

$$\sum |f_i|^2 < \infty \qquad \sum |g_{ik}|^2 < \infty \tag{4}$$

*$f_i, g_{ik}$ ranging over a certain given field, e.g. $\mathbb{C}$ (sometimes only rings, e.g. $\mathbb{Z}$, are admitted; then we are dealing only with modules). We evidently have $H_1^a \subset H_1$.*

*Remark*: One could continue this row of vector spaces in ways which are common practice in, say, *algebraic topology* (see [1], sections 3.1 and 3.2). In this context they are frequently called *chain complexes* (see also [20]). On the other hand, the above vector spaces could also be viewed as *discrete function spaces* over the *node, bond set* with $n_i, d_{ik}$ now representing the elementary *indicator functions*.

Continuing in this spirit we can now introduce two linear maps between $H_0$ and $H_1$, extending the usual *boundary* and *coboundary maps*. On the basis elements they act as follows:

$$\delta: d_{ik} \to n_k \quad \text{hence } b_{ik} \to n_k - n_i \tag{5}$$

$$d: n_i \to \sum_k (d_{ki} - d_{ik}) = \sum_k b_{ki} \tag{6}$$

and are linearly extended. That is; $\delta$ maps the directed bonds $d_{ik}$ onto the terminal node and $b_{ik}$ onto its (oriented) *boundary*, while $d$ maps the node $n_i$ onto the sum of the *ingoing* directed bonds minus the sum of the *outgoing* directed bonds or on the sum of *oriented ingoing* bonds $b_{ki}$.

As was shown in [1] (we later realized that the same definition was already employed in [15]), these definitions lead in fact to a kind of *discrete differential calculus* on $H_0, H_1$; that is, we have

$$df = d\left(\sum f_i n_i\right) = \sum_{k,i} (f_k - f_i) d_{ik}. \tag{7}$$

Combining now the operators $\delta$ and $d$, we can construct the *canonical graph Laplacian*. On the vertex space it reads

$$\delta df = -\sum_i \left(\sum_k f_k - v_i f_i\right) n_i = -\sum_i \left(\sum_k (f_k - f_i)\right) n_i =: -\Delta f \tag{8}$$

where $v_i$ denotes the *node degree* or *valency* defined above and the $k$-sum extends over the nodes adjacent to $n_i$.

*Remark*: Note that there exist several variants in the literature (see, e.g., [9, 17]). Furthermore, many mathematicians employ a different sign convention. We stick, in the following, to the convention used in the mathematical physics literature where $-\Delta$ is the positive(!) operator.

This *graph Laplacian* is intimately connected with another important object, employed in algebraic graph theory, i.e. the *adjacency matrix $A$* of a graph, its entries, $a_{ik}$, having the value 1

if the nodes $n_i$, $n_k$ are connected by a bond and 0 elsewhere. If the graph is *undirected* (but orientable), the relation between $n_i$ and $n_k$ is *symmetric*, i.e.

$$a_{ik} = 1 \Rightarrow a_{ki} = 1. \tag{9}$$

This has the obvious consequence that in the case the graph is *simple* and *undirected*, $A$ is a symmetric matrix with zero diagonal elements.

*Remark*:  More general $A$'s occur if more general graphs are admitted (e.g. general multigraphs).

With our definition of $\Delta$ it holds that

$$\Delta = A - V \tag{10}$$

where $V$ is the diagonal *degree matrix*, having $v_i$ as diagonal entries. (Note that the other sign convention would lead to $\Delta = V - A$.)

To avoid domain problems we assume from now on that the *node degree* $v_i$ is *uniformly bounded* on the graph $G$, i.e.

$$v_i \leqslant v_{\max} < \infty. \tag{11}$$

Defining $d_{1,2}$ as

$$d_{1,2}: \ n_i \rightarrow \sum d_{ki}, \ \sum d_{ik} \tag{12}$$

respectively, and linearly extended, we get

$$d = d_1 - d_2. \tag{13}$$

Similarly, we make the identification $\delta =: \delta_1$ with

$$\delta_{1,2}: \ d_{ik} \rightarrow n_k, \ n_i. \tag{14}$$

It is noteworthy (but actually not surprising) that $v_i \leqslant v_{\max}$ implies that all the above operators are *bounded* (in contrast to their continuous counterparts, which are typically unbounded). Taking this for granted at the moment, a straightforward analysis yields the following relations:

**Lemma 2.2.**

1. *The adjoint $d^*$ of d with respect to the spaces $H_0$, $H_1^{\mathrm{a}}$ is $2\delta$.*
2. *On the other hand we have for the natural extension of d, $\delta$ to the larger space $H_1$:*

$$\delta_1 = (d_1)^* \qquad \delta_2 = (d_2)^* \tag{15}$$

   *hence*

$$(\delta_1 - \delta_2) = (d_1 - d_2)^* = d^* \neq 2\delta = 2\delta_1. \tag{16}$$

3. *Furthermore it holds*

$$d_1^* d_1 = \delta_1 d_1 = d_2^* d_2 = V: \ n_i \rightarrow v_i n_i \tag{17}$$

   *(with V the vertex degree matrix)*

$$d_1^* d_2 = \delta_1 d_2 = d_2^* d_1 = \delta_2 d_1 = A: \ n_i \rightarrow \sum_{k-i} n_k. \tag{18}$$

   *Similar geometric properties of the graph are encoded in the products coming in reverse order.*

(Here and in the following, $k - i$ means summation over the first index and runs through the set of labels of nodes directly connected with $n_i$.)

$d, d^*$ encode certain geometric information about the graph which can be seen from the following domain and range properties (cf [16]; for corresponding results in the more traditional approach, see [8], p 24ff).

**Theorem 2.3.** *Let the graph be connected and finite, $|\mathcal{V}| = n$, then*

$$\dim(\mathrm{Rg}(d^*)) = n - 1 \tag{19}$$

$$\dim(\mathrm{Ker}(d^*)) = \sum_i v_i - (n - 1). \tag{20}$$

*With* $\dim(H_1) = \sum_i v_i \dim\left(H_1^{\mathrm{a}}\right) = \frac{1}{2}\dim(H_1)$, *we have*

$$\mathrm{codim}(\mathrm{Ker}(d^*)) = \dim(\mathrm{Rg}(d)) = n - 1. \tag{21}$$

*We see that both* $\mathrm{Rg}(d^*)$ *and* $\mathrm{Rg}(d)$ *have the same dimension* $(n - 1)$.

**Remark 2.4.** In the case the graph has, say, $c$ components, the above results are altered in an obvious way; we have, for example,

$$\dim(\mathrm{Rg}(d^*)) = n - c. \tag{22}$$

In the literature $\mathrm{Ker}(d^*)$ is called (for obvious reasons) the *cycle subspace* (cf e.g. [8]). On the antisymmetric subspace $H_1^{\mathrm{a}}$ we have $d^* = 2\delta$ and $\delta(b_{ik}) = n_k - n_i$. Choosing now a *cycle*, given by its sequence of consecutive vertices $n_{i_1}, \ldots, n_{i_k}; n_{i_{k+1}} := n_{i_1}$, we have

$$d^* \left( \sum b_{i_l i_{l+1}} \right) = 2 \sum \left( n_{i_{l+1}} - n_{i_l} \right) = 0 \tag{23}$$

that is, vectors of this kind lie in the kernel of $d^*$.

We will now provide quantitative lower and upper bounds for the respective norms of the occurring operators. We have

$$\|df\|^2 = \sum_{ik} |(f_k - f_i)|^2 = \sum_i v_i |f_i|^2 + \sum_k v_k |f_k|^2 - \sum_{i \neq k} (\overline{f_k} f_i + \overline{f_i} f_k)$$

$$= 2 \sum_i v_i |f_i|^2 - 2 \sum_{i \neq k} \overline{f_k} f_i \tag{24}$$

which can be written as

$$\|df\|^2 = 2((f|Vf) - (f|Af)) = (f| -2\Delta f) \tag{25}$$

and shows the close relationship of the norm of $d$ with the *expectation values* of the *adjacency* and *degree matrix* or the *graph Laplacian*. That is, norm estimates for, say, $d$, are derived in a natural manner from the corresponding estimates for $A$ or $-\Delta$. With

$$\|df\|^2 = (f|d^*df) = (f| -2\Delta f) \tag{26}$$

we have

$$0 \leqslant d^*d = -2\Delta \qquad \text{and} \qquad \|d\|^2 = \sup_{\|f\|=1} (f| -2\Delta f) = \|-2\Delta\|. \tag{27}$$

Furthermore, via

$$0 < \sup_{\|f\|=1} (f| -2\Delta f) \leqslant 2v_{\max} + 2 \sup_{\|f\|=1} |\langle f|af \rangle| \tag{28}$$

we get

$$\|-\Delta\| \leqslant v_{\max} + \|A\|. \tag{29}$$

*Remark*: We want to mention that we are using the usual *operator norm* also for the matrices (in contrast to most of the matrix literature), which is also called the *spectral norm*. It is unique in so far as it coincides with the so-called *spectral radius* (cf e.g. [29, 30]), that is

$$\|A\| := \sup\{|\lambda|; \lambda \in \text{spectr}(A)\}. \tag{30}$$

In the first step we give upper and lower bounds for the operator norm of the adjacency matrix $A$, both in the finite- and in the infinite-dimensional cases. There are various proofs available of varying degree of generality (see, e.g., [10, 16, 17]) to which we refer the reader. In the following, we give only the final results. Note, however, that the transition from finite to infinite graphs is far from straightforward, as in some of the necessary technical steps entirely new methods are needed.

**Theorem 2.5** (norm of $A$). *With the adjacency matrix $A$ finite or infinite and a finite $v_{\max}$ we have the following result (a certain fixed labelling of the nodes being assumed):*

$$\limsup n^{-1} \sum_{i=1}^{n} v_i^{(n)} \leqslant \limsup \|A_n\| = \|A\| = \sup\{|\lambda|; \lambda \in \text{spectr}(A)\} \leqslant v_{\max}. \tag{31}$$

*Here $A_n$ are the adjacency matrices for the induced sub-graphs, living over the first n labelled nodes, $v_i^{(n)}$ is the corresponding induced (and n-dependent) node degree.*

As a byproduct we have the important lemma.

**Lemma 2.6.** *The adjacency matrices $A_n$ converge strongly to $A$ and we have in particular $\|A_n\| \nearrow \|A\|$.*

(For a proof of the latter result see [10, 16]).

**Remark 2.7.** To prove strong convergence of operators is of some relevance for the limit behaviour of spectral properties of the operators $A_n, A$. That is, (cf e.g. [31], section VIII.7), we have in that case ($A_n, A$ self-adjoint and uniformly bounded) $A_n \rightarrow A$ in the strong resolvent sense, which implies that the spectrum of the limit operator $A$ cannot suddenly expand, i.e.

$$\lambda \in \text{spec}(A) \Rightarrow \exists \lambda_n \in \text{spec}(A_n) \qquad \text{with} \quad \lambda_n \rightarrow \lambda \tag{32}$$

and for $a, b \notin \text{spec}_{pp}(A)$

$$P_{(a,b)}(A_n) \rightarrow P_{(a,b)}(A) \text{ strongly.} \tag{33}$$

To test the effectiveness of the upper and lower bounds given above, we apply them to a non-trivial model recently discussed in [32], i.e. the infinite *binary tree* with *root $n_0$*, where $v_0$ is 2 and $v_i$ equals 3 for $i \neq 0$. The authors show (among other things) that the spectrum consists of the interval $[-2\sqrt{2}, 2\sqrt{2}]$, i.e. $\|A\| = 2\sqrt{2}$, $v_{\max}$ is 3; we have to calculate $\limsup \frac{1}{n} \sum_1^n v_i$. For simplicity, we choose a subsequence so that $n := n(N)$ with $N$ denoting the $N$th level (consisting of $2^N$ nodes) of the tree starting from the root $n_0$. Note that in the corresponding *induced sub-graph $G_N$* the *boundary nodes* placed in the $N$th level have only one *node degree* with respect to $G_N$ but three when viewed as nodes in the full tree.

We then have

$$n = \sum_{k=1}^{N} 2^k \qquad \sum_{i=0}^{n(N)} v_i = 2 + 3 \sum_{k=1}^{N-1} 2^k + 2^N = 3 \sum_{k=0}^{N} (2^k - 2)(2^N - 1). \tag{34}$$

Hence

$$\lim_{n(N)} 1/n(N) \sum_{i=0}^{n(N)} = 3 - 2 \lim_{N} \left( \sum_{0}^{N} 2^{k-N} \right)^{-1} = 2. \tag{35}$$

That is, our general estimate implies $2 \leqslant \|A\| \leqslant 3$, which is not so bad.

## 2.2. Arbitrary directed graphs

If we deal with the general directed graphs, we have both ingoing and outgoing edges at each node but in general they no longer occur in a symmetric way. But nevertheless, most of our concepts and tools, developed in the foregoing subsection, do still exist. The definitions of $d$ and $d^*$ are unaltered. As each edge, $d_{ik}$, is an *outgoing* edge for node $n_i$ and an *ingoing* edge for node $n_k$, the same expression holds for $df$, i.e.

$$df = \sum_{i,k}(f_i - f_k)d_{ik} \qquad \text{with} \quad f = \sum_i f_i n_i \tag{36}$$

(the sum of course extends only over those directed edges which do exist in the directed graph; in particular, each edge, $d_{ik}$, is only counted once in the sum as an outgoing edge with respect to the label $i$.) Furthermore, the notions of $d_{1,2}$ and $\delta_{1,2}$ remain the same, mapping nodes or edges on ingoing edges, outgoing nodes and vice versa. We again have

$$\delta_{1,2} = d^*_{1,2} \qquad \text{and} \qquad (\delta_1 - \delta_2) = d^* = (d_1 - d_2). \tag{37}$$

We can now calculate $d^*_1 d_1$ and $d^*_2 d_2$ and get

$$d^*_1 d_1(n_i) = v^{\text{in}}_i n_i \qquad d^*_2 d_2(n_i) = v^{\text{out}}_i n_i \tag{38}$$

with $v^{\text{in,out}}_i$ the *in-, out-degree* of the node $n_i$. In the same way we can calculate

$$\delta_1 d_2(n_i) = \sum_{k-i,\text{out}} n_k \qquad \delta_2 d_1(n_i) = \sum_{k-i,\text{in}} n_k. \tag{39}$$

This yields

$$d^*d = (\delta_1 - \delta_2)(d_1 - d_2) = (V^{\text{in}} + V^{\text{out}}) - (A^{\text{in}} + A^{\text{out}}) \tag{40}$$

where the occurring operators on the rhs are the *in-, out-vertex degree matrices and in-, out-adjacency matrices*, respectively.

In general, the individual in-, out-adjacency matrices are non-symmetric. Their sum, however, is symmetric (and is, in the case of a symmetric graph, twice the adjacency matrix of the undirected graph, i.e. $2A$). One can now again define a (positive, i.e. symmetric) Laplace operator for a non-symmetric (directed) graph, that is,

**Conclusion 2.8**

$$-\Delta := d^*d = (V^{\text{in}} + V^{\text{out}}) - (A^{\text{in}} + A^{\text{out}}) =: V_d - A_d \tag{41}$$

(Note the factor 2 now missing in front of $\Delta$!)

## 3. The spectral triplet on a general (directed or undirected) graph

We begin by making some remarks on various concepts in use in the more recent literature. We note that our version of a Dirac operator (defined below) intertwines node-vectors and bond-vectors while in other examples it maps node to node functions. Our bond-functions have (in some sense) the character of *cotangential vectors*, while in other approaches derivatives of functions are interpreted as *tangent vectors*. In our view, the latter point of view is effective only in certain classes of highly regular models (e.g. lattices) where one has global directions, which becomes cumbersome for general graphs. We developed this latter approach a little bit in section 3.3 of [1] and showed how these cotangent and tangent vectors can be mapped into each other. We think that, on the other hand, our framework is more flexible in the general case. This holds, in particular, for our Dirac operator, which encodes certain geometric properties of the underlying discrete 'manifold'.

The Hilbert space we will use in the following is

$$H = H_0 \oplus H_1. \tag{42}$$

The *natural representation* of the function algebra $\mathcal{F}$ (consisting of the bounded node functions)

$$\left\{ f; \, f \in \mathcal{C}^\infty \text{ i.e. } \sup_i |f_i| < \infty \right\} \tag{43}$$

on $H$ by bounded operators is given by

$$H_0: \, ff' = \sum f_i f_i' n_i \qquad \text{for} \quad f' \in H_0 \tag{44}$$

$$H_1: \, f \sum g_{ik} d_{ik} := \sum f_i g_{ik} d_{ik}. \tag{45}$$

From previous work [1] we know that $H_1$ also carries a right-module structure, given by

$$\sum g_{ik} d_{ik} f := \sum g_{ik} f_k d_{ik}. \tag{46}$$

(For convenience, we do not distinguish notationally between elements of $\mathcal{F}$ and their Hilbert space representations.)

*Remark*: The same bimodule structure and the Dirac operator defined below were already employed in [15], p 414ff.

An important object in various areas of modern analysis on manifolds or in Connes' approach to non-commutative geometry is the so-called *Dirac operator D* (or rather, a certain version or variant of its classical counterpart; for the wider context, see e.g. [5, 33–35]). As $D$ we will take in our context the operator

$$D := \begin{pmatrix} 0 & d^* \\ d & 0 \end{pmatrix} \tag{47}$$

acting on

$$H = \begin{pmatrix} H_0 \\ H_1 \end{pmatrix} \tag{48}$$

with

$$d^* = (\delta_1 - \delta_2). \tag{49}$$

Note, however, that there may exist, in general, several possibilities to choose such an operator. On the other hand, we consider our personal choice to be very natural from a geometrical point of view.

**Lemma 3.1.** *There exists in our scheme a natural chirality or grading operator $\chi$ and an antilinear involution J, given by*

$$\chi := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{50}$$

*with*

$$[\chi, \mathcal{F}] = 0 \qquad \chi D + D \chi = 0 \tag{51}$$

*and*

$$J: \begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \overline{x} \\ \overline{y} \end{pmatrix} \tag{52}$$

*so that*

$$J \cdot f \cdot J = \bar{f}. \tag{53}$$

These are some of the ingredients which establish what Connes calls a *spectral triple* (cf e.g. [23, 24]). We do not want, however, to introduce the full machinery at the moment as our scheme has an independent geometric meaning of its own. Note, in particular, what we are saying below about (non) compactness of various operators in observation 3.3.

**Definition 3.2** (Spectral triplets). *As* spectral triplet *on a general graph we take*

$$(H, \mathcal{F}, D). \tag{54}$$

In our general framework, we obtained in a relatively straightforward manner a Hilbert space as the direct sum of the *node space* (a function space) and the *bond space* (resembling the set of cotangent vectors) and a Dirac operator which emerged naturally as a type of square root of the Laplacian.

On the other hand, if one studies simple models as, for example, in [25–27], other choices are possible. In [25, 26], where the one-dimensional lattice was studied, the *symmetric difference operator* was taken as Dirac operator. In [27] the one-dimensional lattice was assumed to be directed (i.e., in our notation, only $d_{i,i+1}$ were present) and the Dirac operator was defined (somewhat *ad hoc*) as a certain self-adjoint 'doubling' of the (one-sided, i.e. non-symmetric) adjacency matrix. As we will show below, this latter model fits naturally in our general approach which includes both directed and undirected graphs. All these Dirac operators are different and hence it is no wonder that they lead to different consequences (see below). It is our opinion that, in the end, an appropriate choice has to be dictated through physical intuition. Nevertheless, this apparent non-uniqueness should be studied more carefully.

As can be seen from the above discussion, the connection with the graph Laplacian is relatively close since for a symmetric graph, for example, we have

$$D^2 = \begin{pmatrix} d^*d & 0 \\ 0 & dd^* \end{pmatrix} \tag{55}$$

and

$$d^*d = -2\Delta \tag{56}$$

$dd^*$ is the corresponding object on $H_1$. (In the vector analysis of the continuum the two entries correspond to divgrad and graddiv, respectively.)

In the original approach of Connes, great emphasis was laid on the compactness of operators like the inverse of the Dirac operator and it is part of the general definition of a *spectral triple*. As discrete spaces of the kind we are studying are non-trivial examples of non-commutative spaces, it is interesting that we can easily test whether or not this assumption is fulfilled in our particular setting. As may be expected, for graphs with globally bounded node degree, we have the following result:

**Observation 3.3.** *Note that all our operators are bounded, the Hilbert space is (in general) infinite dimensional; hence there is no chance to have, for example, $(D - z)^{-1}$ or $(D^2 - z)^{-1}$ compact. At the moment we are sceptical whether or not this latter phenomenon dissappears generically if the vertex degree is allowed to become infinite. There are some results on spectra of random graphs which seem to have a certain bearing on this problem (cf e.g. [28]).*

In the following sections, we introduce and calculate the so-called Connes-distance functional and compare it, among other things, with the ordinary graph distance. In doing this, we have to calculate the commutator $[D, f]$ applied to an element $f' \in H_0$. We have

$$(df)f' = \sum_{ik} (f_k f'_k - f_i f'_i) \, d_{ik} \tag{57}$$

$$(fd)f' = \sum_{ik} f_i (f'_k - f'_i) \, d_{ik} \tag{58}$$

hence

$$[D, f]f' = \sum_{ik}(f_k - f_i)f_k' d_{ik}. \tag{59}$$

On the other hand, the right-module structure allows us to define $df$ as an operator on $H_0$ via

$$dff' = \left(\sum_{ik}(f_k - f_i)d_{ik}\right)\left(\sum_k f_k' n_k\right) = \sum_{ik}(f_k - f_i)f_k' d_{ik} = [D, f]f'. \tag{60}$$

In the next step, we define $df$ as an operator on $H_1$ which is not as natural as on $H_0$. We define

$$df|_{H_1}: d_{ik} \to (f_i - f_k)n_k \tag{61}$$

and it is linearly extended. A short calculation shows

$$df|_{H_1} = -(d\bar{f}|_{H_0})^* = [d^*, f] \tag{62}$$

with

$$[d^*, f]g = d^*(fg) - fd^*g \qquad g \in H_1. \tag{63}$$

This then has the following desirable consequence:

**Conclusion 3.4.** *With the above definitions the representation of $df$ on $H$ is given by*

$$df|_H = \begin{pmatrix} 0 & df|_{H_1} \\ df|_{H_0} & 0 \end{pmatrix} = \begin{pmatrix} 0 & -(d\bar{f}|_{H_0})^* \\ df|_{H_0} & 0 \end{pmatrix} \tag{64}$$

*and it immediately follows*

$$df|_H = \begin{pmatrix} 0 & [d^*, f] \\ [d, f] & 0 \end{pmatrix} = [D, f]. \tag{65}$$

## 4. The Connes-distance function on graphs

From the general theory of operators on Hilbert spaces we know that

$$\|T\| = \|T^*\|. \tag{66}$$

Hence

**Lemma 4.1.**

$$\|[d, f]\| = \|[d, \bar{f}]\| = \|[d^*, f]\| \tag{67}$$

*and*

$$\|[D, f]\| = \|[d, f]\|. \tag{68}$$

**Proof.** The left part of (67) is shown below and is a consequence of formula (74); the right identity follows from (66). With

$$X := \begin{pmatrix} x \\ y \end{pmatrix} \tag{69}$$

and $T_1 := [d, f]$, $T_2 := [d^*, f]$, the norm of $[D, f]$ is

$$\|[D, f]\|^2 = \sup\{\|T_1 x\|^2 + \|T_2 y\|^2; \|x\|^2 + \|y\|^2 = 1\}. \tag{70}$$

Normalizing now $x, y$ to $\|x\| = \|y\| = 1$ and representing a general normalized vector $X$ as

$$X = \lambda x + \mu y \qquad \lambda, \mu > 0 \quad \text{and} \quad \lambda^2 + \mu^2 = 1 \tag{71}$$

we get

$$\|[D, f]\|^2 = \sup\{\lambda^2 \|T_1 x\|^2 + \mu^2 \|T_2 y\|^2; \|x\| = \|y\| = 1, \lambda^2 + \mu^2 = 1\} \quad (72)$$

where now $x$, $y$ can be varied independently of $\lambda$, $\mu$ in their respective admissible sets, hence

$$\|[D, f]\|^2 = \sup\{\lambda^2 \|T_1\|^2 + \mu^2 \|T_2\|^2\} = \|T_1\|^2 \quad (73)$$

as a consequence of equation (67). $\qquad\square$

We see that in calculating $\|[D, f]\|$ we can restrict ourselves to the simpler expression $\|[d, f]\|$. We infer from the above calculations ($x \in H_0$),

$$\|dfx\|^2 = \sum_i \left( \sum_{j=1}^{v_i} |f_i - f_{k_j}|^2 |x_i|^2 \right) \quad (74)$$

and the corresponding expression for a directed graph with $v_i$ replaced by $v_i^{\text{out}}$. Abbreviating

$$\sum_{j=1}^{v_i} |f_{k_j} - f_i|^2 =: a_i \geqslant 0 \quad (75)$$

and calling the supremum over $i$ $a_s$, it follows

$$\|dfx\|^2 = a_s \left( \sum_i \frac{a_i}{a_s} |x_i|^2 \right) \leqslant a_s \quad (76)$$

for $\|x\|^2 = \sum_i |x_i|^2 = 1$.

On the other hand, choosing an appropriate sequence of normalized basis vectors $e_v$ so that the corresponding $a_v$ converge to $a_s$ we get

$$\|dfe_v\|^2 \to a_s. \quad (77)$$

Hence we have

**Theorem 4.2.**

$$\|[D, f]\| = \sup_i \left( \sum_{j=1}^{v_i} |f_{k_j} - f_i|^2 \right)^{1/2}, \sup_i \left( \sum_{j=1}^{v_i^{\text{out}}} |f_{k_j} - f_i|^2 \right)^{1/2} \quad \text{respectively.} \quad (78)$$

The *Connes-distance functional* between two nodes $n$, $n'$ is now defined as follows:

**Definition 4.3** (Connes-distance function).

$$\text{dist}_C(n, n') := \sup\{|f_{n'} - f_n|; \|[D, f]\| = \|df\| \leqslant 1\}. \quad (79)$$

We would like to note that Davies in [15] introduced several metrics on graphs, which have been motivated, as he remarks, by his study of heat kernels on Riemannian manifolds. What he calls metric $d_3$, is related to the rhs of the equation in theorem 4.2 (he uses slightly different Hilbert spaces). Then he shows in a longer proof that this metric is identical to another one, $d_4$, which corresponds to the lhs of the equation in the above theorem. In our approach, however, the content of theorem 4.2 is derived in a relatively transparent and straightforward way.

**Remark 4.4.** It is easy to prove that this defines a metric on the graph.

**Corollary 4.5.** *It is sufficient to vary only over the set* $\{f; \|df\| = 1\}$.

**Proof.** This follows from

$$|f_k - f_i| = c|f_k/c - f_i/c| \qquad c = \|df\| \tag{80}$$

and

$$\|d(f/c)\| = c^{-1}\|df\| = 1 \tag{81}$$

with $c \leqslant 1$ in our case. $\qquad\qquad\square$

In general, it turns out to be a non-trivial task to calculate this distance on an arbitrary graph as the nature of the above constraint is quite subtle. The underlying reason is that the constraint is, in some sense, inherently *non-local*. As $f$ is a function, the difference, $f_{n'} - f_n$, has to be the same independent of the path we follow, connecting $n'$ and $n$. However, in a typical optimization process one usually deals with the individual jumps, $f_k - f_i$, between neighbouring points along some path. Then it is not clear at all whether or not these special choices of jumps along such a path can be extended to a global function without violating the overall constraint on the expression in theorem 4.2. Nevertheless we think the above closed form is a solid starting point for the calculation of $\mathrm{dist}_C$ on various classes of graphs or lattices. We illustrate this by proving some *a priori* estimates concerning this distance function and by evaluating it for some examples.

*4.1. Some general estimates*

Having an admissible function $f$ so that $\sup_i \left( \sum_{k=1}^{v_i} |f_k - f_i|^2 \right)^{1/2} \leqslant 1$ implies that, taking a *minimal path* $\gamma$ from, say, $n$ to $n'$, the jumps $|f_{v+1} - f_v|$ between neighbouring nodes along the path have to fulfil

$$|f_{v+1} - f_v| \leqslant 1 \tag{82}$$

and, *a fortiori*, have to be strictly smaller than 1 in the general situation.

On the other hand, the Connes distance can only become identical to the ordinary distance $d(n, n')$ if there exists a sequence of admissible node functions with all these jumps approaching the value 1 along such a path, which is however impossible in general, as can be seen from the structure of the constraint on the expression in theorem 4.2. Only in this case does one get

$$\left| \sum_\gamma (f_{v+1} - f_v) \right| \to \sum_\gamma 1 = \mathrm{length}(\gamma). \tag{83}$$

We formulate this observation as follows:

**Lemma 4.6** (Connes distance). *Within our general scheme one has the following inequality:*

$$\mathrm{dist}_C(n, n') \leqslant d(n, n'). \tag{84}$$

*By the same token one can prove that* $\mathrm{dist}_C$ *between two nodes is bounded by the corresponding Connes distance calculated for the (one-dimensional) sub-graph formed by a minimal path connecting these nodes, i.e.*

$$\mathrm{dist}_C(n, n') \leqslant \mathrm{dist}_C(\textit{minimal path})(n, n'). \tag{85}$$

The reason is that one has more *admissible functions* at one's disposal for a sub-graph. With $G'$ a connected sub-graph of $G$, the set of admissible function, $S_{G'}$, on $G'$ contains the restrictions of the functions of the corresponding set, $S_G$, belonging to $G$, as each restriction to $G'$ of a member belonging to $S_G$ lies in $S_{G'}$. Hence the supremum is, in general, larger on $S_{G'}$. The distance along such a path, however, can be rigorously calculated (see the discussion of some examples below) and is for non-neighbouring nodes markedly smaller than the ordinary graph distance. From what we have said we can also infer the following corollary.

**Corollary 4.7.** *With $G'$ a connected sub-graph of $G$ it holds (with $n, n' \in V' \subset V$)*

$$\text{dist}_C(n, n'; G) \leqslant \text{dist}_C(n, n'; G'). \tag{86}$$

One can also give sufficient criteria for $\text{dist}_C(n, n') < d(n, n')$. The cases of undirected and directed graphs have to be treated a little bit differently.

**Lemma 4.8.** *Let $G$ be an undirected graph and $\gamma$ a minimal path of length $l > 1$, connecting $n, n'$. There is at least one node, $n^*$, belonging to $\gamma$, having node degree $\geqslant 2$ (as there are at least two consecutive edges, belonging to $\gamma$). If*

$$\text{dist}_C(n, n') = l = d(n, n') \tag{87}$$

*all the individual jumps along $\gamma$ have to be $1$. But then the corresponding function cannot be admissible at $n^*$. Hence*

$$\text{dist}_C(n, n') < d(n, n'). \tag{88}$$

*Let $G$ now be a directed graph and let there exist two different paths, $\gamma, \gamma'$, of equal lengths, $l > 1$, connecting $n, n'$. Again there exists a node $n^*$ on $\gamma$ so that it is incident with two edges, one belonging to $\gamma$, the other to $\gamma'$. Along both edges the jumps have to be $1$ and again the admissibility of the corresponding function is violated. Again we can conclude*

$$\text{dist}_C(n, n') < d(n, n'). \tag{89}$$

*Remark*: The latter situation will be discussed below in the example of the directed $\mathbb{Z}^2$-lattice.

We remarked above that the calculation of the Connes distance on graphs is, to a large extent a, continuation problem for admissible functions, defined on sub-graphs. Then the following question poses itself. For what classes of graphs and/or sub-graphs do we have an equality in the above corollary? We start from a given graph $G_0 = (V_0, E_0)$ and then add new nodes and bonds, yielding a new graph $G' = (V', E')$. We consider two fixed nodes $n_0, n'_0$ in $G_0$.

**Assumption 4.9.** *We assume that the above process does not create new paths between nodes belonging to $G_0$. In other words, the paths, connecting $n_0$ and $n'_0$ are contained in $G_0$.*

**Lemma 4.10.** *Under this assumption, each admissible function on $G_0$ can be extended to an admissible function on $G'$.*

**Proof.** In the first step, we construct the set of *nearest neighbours*, $V_1 \backslash V_0$ in $V' \backslash V_0$ relative to $V_0$. Each new node in $V_1 \backslash V_0$ has a unique nearest neighbour in $V_0$ since otherwise there would exist a new path between these two nodes lying in $G_0$. With $n \in V_1 \backslash V_0$ we extend an admissible function on $G_0$ as follows:

$$f_n := f_{n_0} \qquad n_0 \text{ is the unique nearest neighbour in } V_0. \tag{90}$$

This extended function is an admissible function on $G_1 := (V_1, E_1)$. Note however that, by assumption, there do not exist bonds in $E_1$, connecting nodes in $V_1 \backslash V_0$. We can now continue this process until we arrive at the graph $G'$. $\qquad\square$

By the same token we see that

$$\text{dist}_C(n_0, n'_0; G_1) = \text{dist}_C(n_0, n'_0; G_0). \tag{91}$$

This holds at every intermediate step and we get

**Lemma 4.11.** *Under the above assumption we have*

$$\text{dist}_C(n_0, n_0'; G_0) = \text{dist}_C(n_0, n_0'; G'). \tag{92}$$

**Corollary 4.12.** *If G is a tree, it holds*

$$\text{dist}_C(n; n') = \text{dist}_C(n; n'; \; minimal\; path). \tag{93}$$

**Proof.** In a tree there exists, by definition, at most one path, connecting two nodes. We can take this path as the connected sub-graph $G_0$ and make the above extension, since $G$ and $G_0$ fulfil the assumption. $\qquad\square$

The graphs so constructed are, however, rather special, consisting, so to speak, of a start graph plus some added hair.

We have given above sufficient conditions for

$$\text{dist}_C(n, n'; G_0) = \text{dist}_C(n, n'; G) \tag{94}$$

with $G$ an extension of $G_0$ and $n, n' \in V_0$. We show now that the emergence of too short new paths represents the obstruction for such a result to hold in general.

So let $G_0$ be a graph again and assume the existence of two nodes, $n, n'$, in $V_0$ with

$$\text{dist}_C(n, n'; G_0) > l \in \mathbb{N}. \tag{95}$$

We extend $G_0$ to some $G$ by adding new nodes and edges. We know that for admissible functions the elementary jumps along an edge have to fulfil $|f_i - f_k| \leqslant 1$. If there exists a new path $\gamma$ in $G$, connecting $n, n'$, with

$$\text{length}(\gamma) \leqslant l \tag{96}$$

we can conclude that for each admissible function on $G$ it must hold

$$|f(n) - f(n')| \leqslant l. \tag{97}$$

Hence we have

**Lemma 4.13.** *If two nodes in $G_0$ have $\text{dist}_C(n, n'; G_0) > l$ and if there exists a path $\gamma$ in $G$, connecting $n, n'$ and having length $l \leqslant l$, it necessarily holds that*

$$\text{dist}_C(n, n'; G) \leqslant l < \text{dist}_C(n, n'; G_0). \tag{98}$$

Up to now we have derived upper bounds on $\text{dist}_C$ relative to $\text{dist}_C$ on sub-graphs or the canonical graph distance $d(n, n')$. In the following, we will derive a quite efficient lower bound. This is done by defining a particular admissible function, depending on an arbitrary base node, $n_0$. As the admissible function we choose the canonical graph distance, starting from the arbitrary but fixed node, $n_0$, divided by the local vertex degree; that is,

$$f_{n_0}(n) := (v_n)^{-1/2} d(n_0, n) \qquad f_{n_0}(n) := \left(v_n^{(\text{out})}\right)^{-1/2} d(n_0, n) \qquad f_{n_0}(n_0) = 0 \tag{99}$$

for undirected and directed graphs, respectively.

From our general results we have

$$\|df\| = \sup_i \left( \sum_{j=1}^{v_i} |f_{k_j} - f_i|^2 \right)^{1/2} \tag{100}$$

or $v_i$ replaced by $v_i^{(\text{out})}$. Inserting the above particular function we get

$$\|df\| \leqslant 1 \tag{101}$$

as each term, $|f_{k_j} - f_i|$, is either 0 or 1 (depending on whether the distance to the base point remains constant or changes by $\pm 1$).

**Lemma 4.14.** *The functions $f_{n_0}(n)$, $n_0$ an arbitrary node in $G$, are admissible.*

With two arbitrary nodes $n$, $n'$ in $G$ we take $n$ as base point $n_0$, and have

$$f_{n_0}(n') - f_{n_0}(n) = f_{n_0}(n') = (v_{n'})^{-1/2}d(n', n) \tag{102}$$

(as $f_{n_0}(n) = f_{n_0}(n_0) = 0$). As $\text{dist}_C$ is the supremum over admissible functions, we get

**Theorem 4.15.**

$$\text{dist}_C(n, n') \geqslant (v_{(n,n')})^{-1/2} d(n, n') \tag{103}$$

*with $v_{(n,n')}$ the minimum of the (out) vertex degrees at $n$, $n'$, respectively.*

Note that one can of course choose either $n$ or $n'$ as the base point in the definition of the above admissible function.

### 4.2. Examples

The general results derived above should be compared with the results in, e.g., [25–27]. Choosing the symmetric difference operator as the 'Dirac operator' in the case of the one-dimensional lattice, the authors in [25, 26] obtained a distance which is *strictly greater* than the ordinary distance but their choice does not fulfil the above natural constraint given in theorem 4.2. Note, in particular, that our operator $d$ is a map from node- to bond-functions which is not the case in these examples. In [27] the authors employed a symmetric doubling of the non-symmetric adjacency matrix of the one-dimensional directed lattice $\mathbb{Z}^1$ as Dirac operator. With $v_i^{(\text{out})} = 1$ in this example, our above general estimate yields

$$d(n, n') \leqslant \text{dist}_C(n, n') \leqslant d(n, n') \Rightarrow \text{dist}_C(n, n') = d(n, n') \tag{104}$$

that is, we get the same result for our Dirac operator as for the choice made in [27].

We conclude this paper with the discussion of several examples, which show that, in general, it is quite a non-trivial task to calculate $\text{dist}_C$. The first one is a simple warm-up exercise, the second is the one-dimensional undirected lattice $\mathbb{Z}^1$, discussed also by some of the authors mentioned above (treated, however, within their own schemes) and is not so simple. The last is the directed $\mathbb{Z}^2$-lattice, which we do not solve in closed form, but we provide several estimates.

The technique used in approaching some of the problems may be interesting, in general. It turns out that the proper mathematical context, to which our strategy does belong, is the field of *(non)linear programming* or *optimization* (see, e.g., [36] or any other related book). This can be inferred from the structure of the constraints we get. This means that the techniques developed in this field may perhaps be of use in solving such intricate problems.

*First example*: *The square with vertices and edges*

$$x_1 - x_2 - x_3 - x_4 - x_1. \tag{105}$$

Let us calculate the Connes distance between $x_1$ and $x_3$. As the sup is taken over functions, the summation over elementary jumps is (or rather has to be) path independent (this represents a subtle constraint for practical calculations). It is an easy exercise to see that the sup can be found in the class where the two paths between $x_1$, $x_3$ have the *valuations* $(1 \geqslant a \geqslant 0)$

$$x_1 - x_2\!: a \qquad x_2 - x_3\!: (1 - a^2)^{1/2} \tag{106}$$

$$x_1 - x_4\!: (1 - a^2)^{1/2} \qquad x_4 - x_3\!: a. \tag{107}$$

Hence one has to find $\sup_{0 \leqslant a \leqslant 1}(a + \sqrt{1 - a^2})$. Setting the derivative with respect to $a$ to zero, one gets $a = \sqrt{1/2}$. That is,

**Example 4.16** (Connes distance on a square)**.**

$$\text{dist}_C(x_1, x_3) = \sqrt{2} < 2 = d(x_1, x_3). \tag{108}$$

**Remark 4.17.** As $v_i = 2$, our *a priori* estimate in theorem 4.15 is saturated as

$$\text{dist}_C(x_1, x_2) \geqslant (2)^{-1/2} \times 2 = \sqrt{2}. \tag{109}$$

The next example is considerably more complicated.

*Second example: The undirected one-dimensional lattice*

The nodes are numbered by $\mathbb{Z}$. We want to calculate $\text{dist}_C(0, n)$ within our general framework. The calculation will be done in two main steps. In the first part, we make the (in principle quite complicated) optimization process more accessible. For the sake of brevity we state without proof that it is sufficient to discuss real monotonically increasing functions with

$$f(k) = \begin{cases} f(0) & \text{for } k \leqslant 0 \\ f(n) & \text{for } k \geqslant n \end{cases} \tag{110}$$

and we write

$$f(k) = f(0) + \sum_{i=1}^{k} h_i \qquad \text{for } 0 \leqslant k \leqslant n \quad h_i \geqslant 0. \tag{111}$$

The above optimization process then reads

**Observation 4.18.** *Find* $\sup \sum_{i=1}^{n} h_i$ *under the constraint*

$$h_1^2 \leqslant 1 \qquad h_2^2 + h_1^2 \leqslant 1, \ldots, h_n^2 + h_{n-1}^2 \leqslant 1 \qquad h_n^2 \leqslant 1. \tag{112}$$

The simplifying idea is now the following. Let $h := (h_i)_{i=1}^{n}$ be an admissible sequence with all $h_{i+1}^2 + h_i^2 < 1$. We can then find another admissible sequence $h'$ with

$$\sum h_i' > \sum h_i. \tag{113}$$

Hence the supremum cannot be taken on the interior. We conclude that at least some $h_{i+1}^2 + h_i^2$ have to be 1. There is then a minimal $i$ for which this holds. We can convince ourselves that the process can now be repeated for the sub-string ending at $i + 1$. Repeating the argument we can fill up all the entries up to place $i + 1$ with the condition $h_{l+1}^2 + h_l^2 = 1$ and proceeding now upwards we end up with

**Lemma 4.19** *The above supremum is assumed within the subset*

$$h_1^2 \leqslant 1, h_1^2 + h_2^2 = 1, \ldots, h_{n-1}^2 + h_n^2 = 1, h_n^2 \leqslant 1. \tag{114}$$

This concludes the first step.

In the second step we calculate $\sup |f(0) - f(n)|$ on this restricted set. From the above lemma we now have the constraint

$$h_1^2 \leqslant 1 \qquad h_2^2 = 1 - h_1^2 \qquad h_3^2 = h_1^2 \qquad h_4^2 = 1 - h_1^2 \ldots h_n^2 = 1 - h_1^2 \text{ or } h_1^2 \tag{115}$$

depending on $n$ even or odd. This yields

$$\sup |f(0) - f(n)| = \begin{cases} 1 & \text{for } n = 1 \\ (n/2) \sup \left(h_1 + \sqrt{1 - h_1^2}\right) = (n/2)\sqrt{2} & \text{for } n \text{ even} \\ \sup \left([n/2]\left(h_1 + \sqrt{1 - h_1^2}\right) + h_1\right) & \text{for } n \text{ uneven.} \end{cases} \tag{116}$$

In the case where $n$ is even the rhs can be written as $\sqrt{n^2/2} = \sqrt{[n^2/2]}$. In the case where $n$ is odd, we get by differentiating the rhs and setting it to zero,

$$h_1^{\max} = A_n / \sqrt{1 + A_n^2} \qquad \sqrt{1 - (h_1^{\max})^2} = 1/\sqrt{1 + A_n^2} \tag{117}$$

with $A_n = 1 + 1/[n/2]$. We see that for increasing $n$ both terms approach $1/\sqrt{2}$, the result in the even case. Furthermore, we see that the distance is monotonically increasing with $n$ as should be the case for a distance. This yields in the odd case

$$\text{dist}_C(0, n) = \frac{([n/2] + 1) A_n + [n/2]}{\sqrt{1 + A_n^2}} \tag{118}$$

which is a little bit nasty. Both expressions can, however, be written in a more elegant and unified way (this was a conjecture by W Kunhardt, inferred from numerical examples). For $n$ uneven, a short calculation yields

$$[n^2/2] = (n^2 - 1)/2 = \tfrac{1}{2}(n - 1)(n + 1) = 2[n/2]([n/2] + 1) \tag{119}$$

(with the *floor, ceiling* notation the expressions would become even more elegant). With the help of the latter formula the rhs in (118) can be transformed into

$$\frac{([n/2] + 1) A_n + [n/2]}{\sqrt{1 + A_n^2}} = \sqrt{[n^2/2] + 1}. \tag{120}$$

**Conclusion 4.20.** *For the one-dimensional undirected lattice we have*

$$\text{dist}_C(0, n) = \begin{cases} \sqrt{[n^2/2]} & \text{for } n \text{ even} \\ \sqrt{[n^2/2] + 1} & \text{for } n \text{ odd.} \end{cases} \tag{121}$$

**Remark 4.21.** Again comparing the exact result with our lower bound, we find for $n$ even

$$\text{dist}_C(0, n) \geqslant (2)^{-1/2} n = (2)^{1/2} (n/2) \tag{122}$$

that is, the lower bound is again saturated. For $n$ odd we have instead

$$\text{dist}_C(0, n) = (2)^{-1/2} (n^2 + 1)^{1/2} > (2)^{-1/2} n. \tag{123}$$

*Third example: The directed lattice $\mathbb{Z}^2$*
The vertices in $\mathbb{Z}_d^2$ are denoted by $(i, j)$ or $(x, y)$. The edges point from $(i, j)$ to $(i + 1, j)$ and $(i, j + 1)$; hence, $v_i^{\text{out}} = 2$. As the system is translation invariant, it suffices to calculate the Connes distance between nodes $(0, 0)$ and $(x, y)$ with $x, y > 0$.

For nodes lying parallel to the $x$-, $y$-axis, respectively, we have

$$\text{dist}_C(n, n') = d(n, n'). \tag{124}$$

For $x$ or $y = 0$, there is only one minimal path, connecting $(0, 0)$ and $(x, y)$. Therefore, lemma 4.8 does not apply. For, say, $y = 0$, we choose the following *admissible* function:

$$f(x, y) := x \qquad \text{for all} \quad y. \tag{125}$$

We have

$$|f(x, 0) - f(0, 0)| = |x| = d((0, 0), (x, 0)) \tag{126}$$

and can conclude

$$\text{dist}_C((0, 0), (x, 0)) = d((0, 0), (x, 0)) \tag{127}$$

on $\mathbb{Z}_d^2$. The same holds for the $y$-axis.

For nodes with both $x, y \neq 0$, we have more than one minimal path connecting $(0, 0)$ and $(x, y)$. Our lemma then shows that, necessarily,

$$\text{dist}_C(n, n') < d(n, n'). \tag{128}$$

More detailed estimates will be given below.

If we try to really calculate the Connes distance on $\mathbb{Z}_d^2$ for points in *general positions*, the optimization problem becomes quite involved and we will only provide some estimates. The reason is that the constraint equations are of a quite *non-local* nature (compared with the simpler undirected $\mathbb{Z}^1$-lattice) and that, in general, several minimal paths do exist which make the continuation problem quite intricate.

It is easy to see that the canonical graph distance between the points $(0, 0)$ and $(x, y)$ is $|x| + |y|$ and that all minimal paths have the same length. With $x, y > 0$, we conjecture (without giving a proof) that it suffices to restrict the variation to admissible functions with positive jumps in the positive $x$-, $y$-directions and that we can set $f(0, 0) = 0$. A particular minimal path consists of $x$ steps in the $x$-direction followed by $y$ steps in the $y$-direction. We denote (for convenience) the jumps along the $x$-, $y$-axes, respectively, by

$$h_{i0} := f(i, 0) - f(i - 1, 0) \geqslant 0 \; h_{0j} := f(0, j) - f(0, j - 1) \geqslant 0. \tag{129}$$

The optimization problem now reads

**Problem 4.22.** Find $\sup\left(\sum_{i=1}^x h_{i0} + \sum_{j=1}^y h_{0j}\right)$ under the constraints imposed by the admissibility of the corresponding function $f$. Note, however, that the constraints must hold on the full lattice.

From our general result in theorem 4.15, we know that

$$\text{dist}_C((0, 0), (x, y)) \geqslant (2)^{-1/2}(x + y). \tag{130}$$

We can construct an admissible function, $f$, which fulfils

$$|f(0, 0) - f(x, y)| = (2)^{-1/2}(x + y). \tag{131}$$

This can be achieved by setting $f(0, 0) = 0$ and by choosing all $x$-, $y$-jumps equal to $a$ with $2a^2 = 1 \Rightarrow a = (2)^{-1/2}$. This yields the above result.

The question is, whether or not this is already the supremum over the set of admissible functions. We will show that this is not the case by providing another admissible function yielding a bigger value. We choose an admissible function with $x$-jumps equal to $a$ and $y$-jumps equal to $b$ with $a^2 + b^2 = 1$. The admissible function reads

$$f(x, y) = ax + by. \tag{132}$$

The function $f$ takes a *stationary value* at

$$a = (x^2/(x^2 + y^2))^{1/2} \qquad b = (y^2/(x^2 + y^2))^{1/2} \tag{133}$$

yielding the value

$$f(x, y) = (x^2 + y^2)^{1/2}. \tag{134}$$

Assuming, for example, that $x \neq y$, it follows that

$$f(x, y)^2 - ((2)^{-1/2}(x + y))^2 = \tfrac{1}{2}(x^2 + y^2) - xy = \tfrac{1}{2}(x - y)^2 > 0. \tag{135}$$

In other words, we see

**Observation 4.23.** *For the directed $\mathbb{Z}^2$-lattice and $x \neq y$ ($x, y > 0$), we have the estimate*

$$d((0, 0), (x, y)) > \text{dist}_C((0, 0), (x, y)) \geqslant (x^2 + y^2)^{1/2} > (2)^{-1/2}(x + y). \tag{136}$$

# References

[1] Requardt M 1998 Cellular networks as models for Planck-scale physics *J. Phys. A: Math. Gen.* **31** 7997
   (Requardt M 1998 *Preprint* hep-th/9806135)
[2] Requardt M 2000 (Quantum) space-time as a statistical geometry of lumps in random networks *Class. Quantum Grav.* **17** 2029
   (Requardt M 1999 *Preprint* gr-qc/9912059)
[3] Requardt M 2000 Let's call it nonlocal quantum physics *Preprint* gr-qc/0006063
[4] Nowotny Th and Requardt M 1998 Dimension theory on graphs and networks *J. Phys. A: Math. Gen.* **31** 2447
   (Nowotny Th and Requardt M 1997 *Preprint* hep-th/9707082)
[5] Connes A 1994 *NonCommutative Geometry* (New York: Academic)
[6] Harary F ed 1999 *Graph Theory and Theoretical Physics* (New York: Academic)
[7] Strichartz R S 1999 Analysis on fractals *Not. Am. Math. Soc.* **46** 1199
[8] Biggs N 1974 *Algebraic Graph Theory* 1st edn (Cambridge: Cambridge University Press) (1993 2nd edn considerably extended)
[9] Cvetkovic D M, Doob M and Sachs H 1980 *Spectra of Graphs* (New York: Academic)
[10] Mohar B 1982 The spectrum of an infinite graph *Linear Algebr. Appl.* **48** 245
[11] Cvetković D, Doob M, Gutman I and Torĝasev A 1988 *Recent Results in the Theory of Graph Spectra (Ann. Discr. Math. 36)* (New York: North-Holland)
[12] Rieffel M 1999 Metrics on state spaces *Doc. Math.* **4** 559
   (Rieffel M 1999 *Preprint* math.OA/9906151v2)
[13] Weaver N 2000 Lipschitz algebras and derivations II *J. Funct. Anal.* **178** 64
   (Weaver N 1998 *Preprint* math.FA/9807096)
[14] Iochum B, Krajewski Th and Martinetti P 2001 Distances in finite spaces from non-commutative geometry *J. Geom. Phys.* **37** 100
   (Iochum B, Krajewski Th and Martinetti P 1999 *Preprint* heh-th/9912217)
   Paschke M and Sitarz A 1998 Discrete spectral triples and their symmetries *J. Math. Phys.* **39** 6191
[15] Davies E B 1993 Analysis on graphs and non-commutative geometry *J. Funct. Anal.* **111** 398
[16] Requardt M 2000 Graph-Laplacians and Dirac operators on (infinite) graphs and the calculation of the Connes-distance-functional *Preprint* math-ph/0001026
[17] Colin de Verdière Y 1998 *Spectres de Graphes* (Paris: Soc. Math. de France)
   Colin de Verdière Y 1999 Spectre d'operateurs différentiels *Cortona Conf. (June 1997) Symp. Math.* **39** 139
[18] Malozemov L and Teplyaev A 1995 Pure point spectrum on fractal graphs *J. Funct. Anal.* **129** 390
[19] Kostrykin V and Schrader R 1999 Kirchhoff's rule for quantum wires *J. Phys. A: Math. Gen.* **32** 595
[20] Novikov S P 2000 Schrödinger operators on graphs and symplectic geometry *Arnold Proc. Fields Inst. Comm.* vol 24 *Preprint* math-ph/0004013
   Novikov S P 1999 Discrete Schrödinger operators and topology *Preprint* math-ph/9903025 (*Asian. J. Math.* at press)
[21] Loll R 1998 Discrete approaches to quantum gravity *Liv. Rev. Rel.* **1** 13-loll
[22] Ambjorn J, Carfora M and Marzuoli A 1997 *The Geometry of Dynamical Triangulations (Lecture Notes in Physics 50)* (New York: Springer)
[23] Connes A 1996 Gravity coupled with matter *Commun Math. Phys.* **182** 155
   (Connes A 1996 *Preprint* hep-th/9603053)
[24] Krajewski Th 1997 Classification of finite spectral triples *J. Geom. Phys.* **28** 1
   (Krajewski Th 1997 *Preprint* hep-th/9701081)
[25] Bimonte G, Lizzi F and Sporano G 1994 *Phys. Lett.* B **341** 139
[26] Atzmon E 1996 *Lett. Math. Phys.* **37** 341
[27] Dimakis A and Mueller-Hoissen F 1998 *Int. J. Theor. Phys.* **37** 907
   (Dimakis A and Mueller-Hoissen F 1997 *Preprint* q-alg/9707016)
[28] Juhasz F 1978 On the spectrum of a random graph *25 Algebraic Methods in Graph Theory: Coll. Math. Soc. J. Bolayi* p 313
[29] Kato T 1966 *Perturbation Theory of Linear Operators* (New York: Springer)
[30] Achieser N I and Glasmann I M 1968 *Theorie der Linearen Operatoren im Hilbert Raum* (Berlin: Akademie)
[31] Reed M and Simon B 1980 *Methods of Modern Mathematical Physics, Functional Analysis* (New York: Academic)
[32] Allard C and Froese R 1998 A Mourre estimate for a Schrödinger operator on a binary tree *Louisville AMS Meeting (March 1998) Preprint* math-ph/9807007
[33] Landi G 1997 *An Introduction to Noncommutative Spaces and their Geometries (Lecture Notes in Physics, New*

*Series m 51)* (Berlin: Springer)

(Landi G 1997 *Preprint* hep-th/9701078)

[34] Friedrich Th 1997 *Dirac-Operatoren in der Riemannschen Geometrie* (Braunschweig: Vieweg)

[35] Esposito G 1997 Dirac operator and spectral geometry *Preprint* hep-th/9704016

[36] Jungnickel D 1999 *Optimierungsmethoden* (New York: Springer)

[37] Requardt M and Roy S 2001 Quantum space-time as a statistical geometry of fuzzy lumps *Class. Quantum Grav.* **18** 3039

(Requardt M and Roy S 2000 *Preprint* gr-qc/0011076)

[38] Schweizer B and Sklar A 1983 *Probabilistic Metric Spaces* (New York: North-Holland)